Short Questions

1. In general, the MRTS between two inputs depends on the utilization rate of other inputs, i.e., in a production function $f(K,L,E)$, $MRTS_{K,L}$ would in general be a function of E. Show that this is not the case for a Cobb-Douglas production function, i.e., show that the for the production function $q = A\ K^{\alpha}\ L^{\beta}\ E^{\gamma}$, $MRTS_{K,L}$ does not depend on E.

The *MRTS* of capital for labor is given by the ratio of the marginal products of capital and labor. In particular,

$$MRTS_{K,L} = \frac{MP_K}{MP_L}$$

Since

$$MP_K = \alpha\ A\ K^{\alpha-1}\ L^{\beta}\ E^{\gamma}$$

and

$$MP_L = \beta\ A\ K^{\alpha}\ L^{\beta-1}\ E^{\gamma}$$

the expression for the *MRTS* becomes

$$\begin{aligned} MRTS_{K,L} &= \frac{\alpha\ A\ K^{\alpha-1}\ L^{\beta}\ E^{\gamma}}{\beta\ A\ K^{\alpha}\ L^{\beta-1}\ E^{\gamma}} \\ &= \frac{\alpha}{\beta}\ \frac{K^{-1}}{L^{-1}} \\ &= \frac{\alpha}{\beta}\ \frac{L}{K} \end{aligned}$$

Notice that this expression does not depend on the level of energy inputs, *E* (as this drops out of the ratio). Also notice that because the Cobb-Douglas is symmetric with respect to the inputs, a similar result would have been obtained if you had picked any pair of inputs. Thus, for the Cobb-Douglas, the *MRTS* between two inputs does not depend on the utilization level of the other inputs.

2. Consider the production function

$$f(x, z) = x^2 + 2\ z^2$$

A. What is the $MRTS_{x,z}$?

$$MRTS_{x,z} = \frac{MP_x}{MP_y}$$

$$= \frac{2\ x}{4\ z}$$

$$= \frac{1}{2}\ \frac{x}{z}$$

B. What is the elasticity of scale for this production function? Show your work, starting from the definition of the elasticity of scale.

$$\epsilon_{scale} = \left.\frac{\partial \log[f(m\mathbf{y})]}{\partial \log(m)}\right|_{m=1}$$

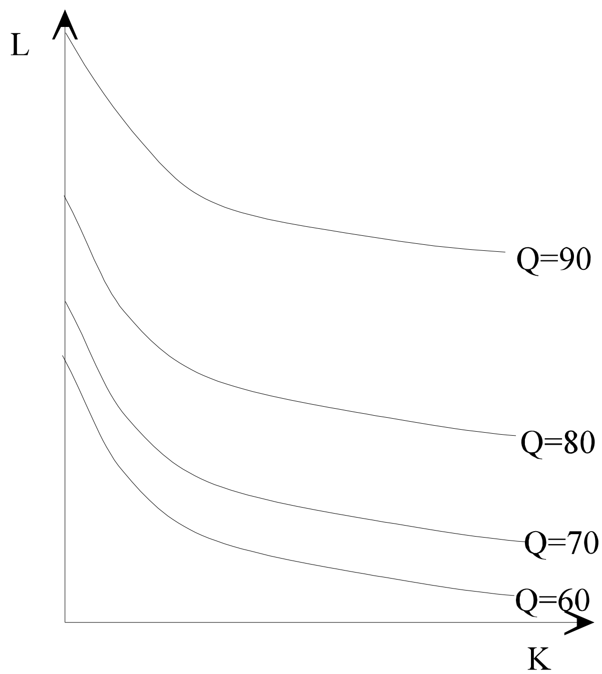
$$= \left.\frac{\partial \log[(m\,x)^2 + 2\ (m\,z)^2]}{\partial \log(m)}\right|_{m=1}$$

$$= \left.\frac{\partial \log[m^2\ (x^2 + 2\ z^2)]}{\partial \log(m)}\right|_{m=1}$$

$$= \left.\frac{\partial [2 \log(m) + \log(x^2 + 2\ z^2)]}{\partial \log(m)}\right|_{m=1}$$

$$= 2$$

3. Consider the production function with isoquants as given in the figure below.



From this figure, does it appear that this production function has increasing or decreasing returns to scale? Explain you answer.

This production function appears to have decreasing returns to scale because isoquants that correspond to the same increment in output are spaced further apart as we move to higher output levels. The gap in the isoquants going from 60 to 70 units of output is smaller than the gap between the 70 and 80 unit isoquants, which in turn is smaller than the gap between the 80 and 90 unit isoquants. Therefore, increasing output by ten units requires progressively more and more inputs, thus returns to scale are decreasing.

Problems

1. Consider the Cobb-Douglas production function for engineers (E) and technicians (T) given by $Q = 10\ E^{0.6}\ T^{0.3}$.

a. If the firm employees an equal number of engineers and technicians, how many technicians will it take to do the work of one engineer? In other words, what is the MRTS of engineers for technicians? [Derive your answer from one of the general expressions for MRTS.]

From the general expression for the MRTS of engineers for technicians, we have:

$$MRTS_{E,T} = \frac{MP_E}{MP_T} = \frac{10\ 0.6\ E^{-0.4}\ T^{0.3}}{10\ 0.3\ E^{0.6}\ T^{-0.7}}$$

which simplifies to

$$MRTS_{E,T} = 2\ \frac{T}{E}$$

Therefore, if the firm has an equal number of engineers and technicians, it will take 2 technicians to do the work of one engineer.

b. What must be the ratio of engineers to technicians employed in this firm so that one technician will be as productive as one engineer?

From the answer of part (a), we know that for the MRTS to be equal to 1, it must be that

$$1 = 2\ \frac{T}{E} \quad \Rightarrow \quad E = 2\ T$$

Therefore, if engineers and technicians are to be equally productive, the firm must employ twice as many engineers as technicians.