Answers to Short Questions

1. What is the economic interpretation of the Lagrange multiplier?

The Lagrange multiplier gives us the change in the value of the objective function per unit of change in the constraint (per unit of relaxing the constraint).

2. How does the substitution method work?

A binding constraint implies that the decision maker has less freedom to choose his actions to maximize his payoff. A binding constraint reduces his degrees of freedom by one. The substitution method works by incorporating this observation directly into the optimization problem: It first solves the constraint for one of the decision variables, which is now expressed as a function of the other decision variables. Then, it substitutes this expression for that decision variable in the objective function. The concentrated objective function now one degree of freedom (one variable) less than the original objective function. In the final step, one maximizes this concentrated objective function with respect to the remaining decision variables. The value of the decision variable that has been "solved out" of the problem is given by using the values of the other variables and the constraint.

Solutions to Problems

1. An accounting firm uses partners and staff to produce an audit. The quality of the audit (as measured by reduction in litigation liability and the likelihood of audit errors) is a function of the composition of the audit team. In particular,

$$r = P^{0.2} S^{0.4}$$

where r is the audit quality, P is the partner-hours devoted to the audit, and S is the staff-hours devoted to the audit. Notice that both partners and staff are essential for audit quality, and that audit quality increases in the amount of either, but at a decreasing rate.

The cost of a partner-hour is 50 while the cost of a staff-hour is 5. The budget for this audit is 5000. How many partner and staff hours will the accounting firm choose to maximize quality subject to this budget constraint?

This is a problem of constrained optimization. The firm chooses P and S to maximize the function

$$r = P^{0.2} S^{0.4}$$

subject to the constraint

$$50 P + 5 S \leq 5000$$

Observe that the objective is increasing in both P and S. Therefore, the audit firm will spend the entire budget on the audit and the constraint will be met with equality, i.e.,

$$50 P + 5 S = 5000$$

The Lagrangian of the problem is given by

$$\mathcal{L} = P^{0.2} S^{0.4} + \lambda (5000 - 50 P - 5 S)$$

The first order conditions of maximization with respect to P, S, and the Lagrange multiplier, λ are

$$\frac{\partial \mathcal{L}}{\partial P} = 0 \quad \Rightarrow \quad 0.2 P^{-0.8} S^{0.4} - 50 \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial S} = 0 \quad \Rightarrow \quad 0.4\ P^{0.2}\ S^{-0.6} - 5\ \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \quad \Rightarrow \quad 5000 - 50\ P - 5\ S = 0$$

One of the "tricks" that often works in obtain a quick solution to such systems is to take the first two equations that involve the Lagrange multiplier, move for each of them the term that has the multiplier on the right hand side, then divide them term by term. The Lagrange multiplier drops out, and we are left with a system of two equations and two unknowns that we can easily solve.

We now apply this method on this problem. The first two first order conditions can be written as

$$0.2\ P^{-0.8}\ S^{0.4} = 50\ \lambda$$

$$0.4\ P^{0.2}\ S^{-0.6} = 5\ \lambda$$

Dividing these equations term by term we get

$$\frac{0.2\ P^{-0.8}\ S^{0.4}}{0.4\ P^{0.2}\ S^{-0.6}} = 10 \quad \Rightarrow$$

$$\frac{2}{4}\ \frac{S}{P} = 10 \quad \Rightarrow$$

$$S = 20\ P \tag{1}$$

This equation and the constraint provide a system of two equations in two unknowns. Substituting this expression for S into the constraint yields

$$5000 - 50\ P - 5\ 20\ P = 0 \quad \Rightarrow$$

$$P = \frac{100}{3} \approx 33.33$$

The above gives us the optimal number of partner-hours for this audit engagement. Plugging this into equation (1) above we get

$$S = \frac{2000}{3} \approx 666.67.$$

This is the optimal number of staff-hours for this audit engagement.

2. Consider a two product firm with a profit function

$$\Pi(q_1, q_2) = -50 + 4\ q_1 - 2\ q_1^2 + 3\ q_2 - q_2^2 - q_1\ q_2$$

where q_1 and q_2 are the output levels of products 1 and 2, respectively. The manufacturing of the two products uses a scarce resource: one unit of product 1 uses one unit of the resource; one unit of product 2 uses two units of the resource. The firm has K units of this scarce resource.

a. Write down the firm's constraint that involves the use of this scarce product.

The constraint is given by

$$q_1 + 2\ q_2 \leq K$$

b. Assume that the constraint is binding, i.e., that K is sufficiently small that all of the scarce resource will be used. Solve the constrained maximization problem of the firm using the substitution method.

If the constraint is binding, then the output levels of products 1 and 2 are linked by the equation

$$q_1 + 2\ q_2 = K$$

Solving for q_1 we obtain

$$q_1 = K - 2\ q_2$$

Substituting for q_1 in the profit function, we can write in terms of only q_2 as:

$$\Pi_c(q_2) = -50 + 4\ (K - 2\,q_2) - 2\ (K - 2\,q_2)^2 + 3\ q_2 - q_2^2 - (K - 2\,q_2)\ q_2$$

Maximizing the profit function with respect to q_2 we obtain (with a liberal use of the product and chain rules of differentiation) the first order condition

$$\frac{d\Pi_c(q_2)}{dq_2} = 0 \quad \Rightarrow \quad -8 + 8\,(K - 2q_2) + 3 - 2\,q_2 + 2\,q_2 - K + 2\,q_2 = 0$$

Simplifying we get

$$-5 + 7\,K - 14\,q_2 = 0$$

Solving for q_2 we obtain

$$q_2 = \frac{1}{2}\,K - \frac{5}{14}$$

This is the optimal output level of product 2. [However, if the value of *K* is smaller than 5/7, then the output level of product 2 will be zero, since negative output is not possible].

Substituting the optimal value for q_2 into the expression for q_1 above, we get

$$q_1 = K - 2\left(\frac{1}{2}K - \frac{5}{14}\right)$$

$$= \frac{5}{7}$$

This is the optimal output level of product 1. [However, if the value of *K* is smaller than 5/7, then the output level of product 1 would be equal to *K*, since a production level of 5/7 would not be feasible.]

c. What is the profit of the firm at the optimal values of q_1 and q_2? [Hint: the answer will be a function of *K*.]

Substituting the optimal values of q_1 and q_2 into the profit function we get

$$\Pi^*(K) = -50 + 4\,\frac{5}{7} - 2\left(\frac{5}{7}\right)^2 + 3\left(\frac{K}{2} - \frac{5}{14}\right) - \left(\frac{K}{2} - \frac{5}{14}\right)^2 - \left(\frac{5}{7}\right)\left(\frac{K}{2} - \frac{5}{14}\right)$$

d. What is the marginal value of the scarce resource to the firm (in terms of increased profit)?

Differentiating $\Pi^*(K)$ with respect to K we get (using the chain rule)

$$\frac{d\,\Pi^*(K)}{d\,K} = 3\ \frac{1}{2} - 2\left(\frac{K}{2} - \frac{5}{14}\right)\frac{1}{2} - \frac{5}{7}\ \frac{1}{2}$$

$$= \frac{3}{2} - \frac{K}{2} + \frac{5}{14} - \frac{5}{14}$$

$$= \frac{3}{2} - \frac{K}{2}$$

e. For what value of K would the resource no longer be scarce, i.e., how high must K be for the firm to choose not to use all of this resource?

Observe that since

$$\frac{d\,\Pi^*(K)}{d\,K} = \frac{3}{2} - \frac{K}{2}$$

profits are increasing for $K < 3$. When K is greater than 3, the profits appear to be decreasing in K. But this appears to be the case because in part (b) we have assumed that the firm will exhaust its entire stock of the resource, i.e., we assumed that

$$q_1 + 2\ q_2 = K$$

However, the actual constraint is

$$q_1 + 2\ q_2 \leq K.$$

In other words, the firm does not have to use the entire stock of the resource. If it happens to have too much, it will just leave it unused.

In particular, the firm will use incremental additions to the stock of the resource as long as it is profitable to do so, that is, as long as

$$\frac{d\,\Pi^*(K)}{d\,K} > 0 \quad \Rightarrow$$

$$\frac{3}{2} - \frac{K}{2} > 0 \quad \Rightarrow$$

$$K < 3 .$$

When the quantity of the resource reaches 3, any addition to it will remain unused.

3. A firm can raise up to 10 million dollars in the financial markets to develop and market a new product. The profits of the firm are given (in million of dollars) by the equation

$$\Pi = 200 \; D^{0.6} \; M^{0.2} - D - M$$

where D is the amount of money spent on development (in millions) and M is the amount of money spent on marketing (in millions). The constraint that the firm can raise up to 10 million for D and M implies that

$$D + M \leq 10$$

Assume that this constraints binds (which is indeed the case), i.e, take as given that .

$$D + M = 10$$

a. What is the Lagrangian expression for this constrained maximization problem?

We first need to assume that the constraint binds, and write it in the form "what the firm has at its disposal minus what it spends." Written this way, the constraint becomes:

$$10 - D - M = 0$$

Then, the Langrangian expression is given by

$$\mathcal{L} = 200 \; D^{0.6} \; M^{0.2} - D - M + \lambda \; (10 - D - M)$$

b. What are the associated First Order Conditions of maximization of the Lagrangian? [Note: there is no need to proceed to the solution of this particular problem, as it is algebraically too tedious.]

They are

$$\frac{\partial \mathcal{L}}{\partial D} = 0 \Rightarrow 0.6\ 200\ D^{-0.4}\ M^{0.2} - 1 - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial M} = 0 \Rightarrow 0.2\ 200\ D^{0.6}\ M^{-0.8} - 1 - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow 10 - D - M = 0$$

c. What would the way the profit function look like if I used the constraint to substitute away one of the two variables?

We can solve the budget constraint for either *D* or *M*. If we solve it for *D*, we get

$$D = 10 - M$$

Using the above expression to replace D from the objective function, we get:

$$\Pi_c = 200\ (10 - M)^{0.6}\ M^{0.2} - 10 + M - M$$

$$= 200\ (10 - M)^{0.6}\ M^{0.2} - 10$$

4. A person's satisfaction (or utility) from playing tennis and golf is given by

$$U = \log(G) + 3\ \log(T)$$

where *G* is the number of hours spent playing golf and *T* is the number of hours spent playing tennis. This person has 10 hours per week to devote to these sports. However, each hour of playing tennis typically entails one hour of waiting for an empty court, thus using up twice the time of actual play. As a consequence, for example, if he spent 2 hours playing golf and 4 hours playing tennis, he would have used up the full 10 hours of his available time.

a. What equation describes this person time constraint?

Since every hour of playing tennis uses up two hours of his available time, and he has 10 hours total in his disposal, the time constraint of this person is

$$G + 2\ T \leq 10$$

Given that satisfaction increases in both *G* and *T*, it is clear that this person will spend all of this available time in one of the two sports, and thus that the constraint will bind. We can then write

$$G + 2\,T = 10$$

b. What is the Lagrangian expression of this constrained maximization problem?

The Lagrangian expression is given by

$$\mathcal{L} = \log(G) + 3\,\log(T) + \lambda\,(10 - G - 2\,T)$$

c. Use this Lagrangian expression to find out what is the satisfaction (or utility) maximizing choice of time to play golf and tennis.

We need to maximize the Lagrangian with respect to *G*, *T*, and λ. The solution will provide the optimal values of *G* and *T*. The first order conditions are given by

$$\frac{\partial \mathcal{L}}{\partial G} = 0 \Rightarrow \frac{1}{G} - \lambda = 0 \Rightarrow \frac{1}{G} = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial T} = 0 \Rightarrow \frac{3}{T} - 2\,\lambda = 0 = \frac{3}{T} = 2\,\lambda$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow 10 - G - 2\,T = 0$$

Dividing the two first equations term by term, we get

$$\frac{\frac{1}{G}}{\frac{3}{T}} = \frac{\lambda}{2\,\lambda} \Rightarrow$$

$$\frac{T}{3\,G} = \frac{1}{2}$$

We need to solve this equation for one of the two variables, and then substitute it into the third equation (the time constraint). Solving for *T*, we get

$$T = \frac{3}{2} G$$

Plugging for *T* into the budget constraint we get

$$10 - G - 2 \frac{3}{2} G = 0 \Rightarrow$$

$$10 - 4 G = 0 \Rightarrow$$

$$G = 2.5$$

This person will play 2.5 hours of golf. Using the equation above the relates *T* with *G*, we can obtain the optimal number of hours of tennis to be

$$T = \frac{3}{2} 2.5 = 7.\frac{5}{2} = 3.75$$

One can readily check that these two time commitments do indeed exhaust the 10 hours available for sports since 2.5 + 2 * 3.75 = 10.

5. An upstart firm has a total expense budget of *B*. This budget can be spent on (i) advertizing the firm's product and (ii) R&D that reduces the production costs. Denote the amount spent on advertizing by *A* and the amount spent on R&D by *RD*. The expense budget, *B*, is determined by the firm's financial backers, and is not in the control of the firm. The firm only controls the values of *A* and *RD*.

Following the decision on advertising and R&D, the firm produces the product at a unit cost of $c = 5 - RD$ and sells it at a unit price of 10. Note that the production costs are **not** financed out of the expense budget *B*. They are incurred by the firm, but are financed out of current revenue. The quantity that the firm sells equals $Q = 5 A$, i.e., the more the firm advertizes, the more units of the product it will be able to sell.

a. Write the firm's profit function in terms of *A* and *RD* (i.e., in terms of the decision variables of the firm). [Note that costs consist of total production costs, the advertising expenditure and the R&D expenditure.]

   The profit function is

$$\Pi = P\ Q(A) - c(RD)\ Q(A) - A - RD$$
$$= 10\ 5\ A - (5 - RD)\ 5\ A - A - RD$$

Multiplying terms out and simplifying we get

$$\Pi = 10\ 5\ A - (5 - RD)\ 5\ A - A - RD$$
$$= 50\ A - 25\ A + 5\ A\ RD - A - RD$$
$$= 24\ A - RD + 5\ A\ RD$$

b. What allocation of funds to advertizing and R&D maximizes the firm's profits? You can solve this problem either with the substitution or the Lagrangian method. Continue to the back side of the page if necessary.

The budget constraint of the firm is

$$B = A + RD$$

Therefore, the Lagrangian is

$$\mathcal{L} = 24\ A - RD + 5\ A\ RD + \lambda\ (B - A - RD)$$

The first order conditions of profit maximization are

$$\frac{\partial \mathcal{L}}{\partial A} = 0 \Rightarrow 24 + 5\ RD = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial RD} = 0 \Rightarrow -1 + 5\ A = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow B - A - RD = 0$$

Dividing the first two equations term by term we get

$$\frac{24 + 5\ RD}{-1 + 5\ A} = 1$$

Solving for A, we get

$$24 + 5\,RD = -1 + 5\,A \Rightarrow$$

$$A = 5 + RD$$

Substituting this into the budget constraint we get

$$B - 5 - RD - RD = 0$$

Solving for *RD*, we find that the optimal amount of R&D equals

$$RD = \frac{B}{2} - \frac{5}{2}$$

It immediately follows from the above that the optimal amount of advertising is

$$A = 5 + \frac{B}{2} - \frac{5}{2}$$

$$= \frac{B}{2} + \frac{5}{2}$$

Discussion: Notice that for values of *B* less than 5, the predicted spending on R&D is negative. Since negative spending is not possible, for such values of B the firm would spend all the budget on advertising, and it would only start to spend money on R&D, using the formula above, if the budget exceeds 5.

Some discussion on the optimal solution to this problem is there was no budget constraint

What would the firm do if its expenditure on *A* and *RD* was not limited by the amount of financing? It is easy to see that it would spent as much money on advertising as possible. Even if the spending on R&D were zero, the profit function would equal $\Pi = 25\,A - A$, which is unbounded in *A*! So advertising spending would go to infinity, and so would sales. It then follows that (with so high level of output) it would pay for the firm to increase R&D until unit production costs would go to zero.

The above discussion shows that the budget constraint of the firm will indeed be binding; the firm will not leave any funds unspent.

Note, however, that if someone was to maximize the profit function of the firm subject to no budget constraint, the optimal solution would be finite. In particular,

$$\frac{\partial \Pi}{\partial A} = 0 \Rightarrow 24 + 5\ RD = 0 \quad \Rightarrow \quad RD = 4.8$$

$$\frac{\partial \Pi}{\partial RD} = 0 \Rightarrow -1 + 5\ A = 0 \quad \Rightarrow \quad A = 0.2$$

The seeming contradiction between these mathematical results and the discussion in the paragraph above can be resolved by observing that the mathematical solution above violates the second order conditions of maximization for functions of two variables. In fact, it can readily verified that the profit obtained from the above solution is less than that from, say *A*=100, *RD*=0.