Answers to Short Questions

1. Consider the function $y = z^2 e^{2q}$. What are the first order partial derivatives of this function with respect to z and q?

The first order partial with respect to z is

$$\frac{\partial y}{\partial z} = 2\ z\ e^{2q}$$

while the first order partial derivative with respect to q is

$$\frac{\partial y}{\partial q} = 2\ z^2\ e^{2q}$$

2. A farm is using two major inputs: fertilizer (F) and pesticides (P). Its output of corn (C) is given by the function

$$C = 10 + 4\ \log(1+F) + \log(1+2P)$$

What is the incremental change in output for this farm if it increases both its inputs, with fertilizer increase being 5 times that of the pesticide increase?

The incremental change of output as both inputs change is given by the total differential of the function. To compute the total differential, we must first calculate the partial derivatives of the function.

The partial derivative with respect to F is

$$\frac{\partial C}{\partial F} = \frac{4}{1 + F}$$

The partial derivative with respect to P is

$$\frac{\partial C}{\partial P} = \frac{2}{1 + 2P}$$

The total differential of the function is

$$dC = \frac{4}{1 + F} dF + \frac{2}{1 + 2P} dP$$

where dF is the (incremental) change in fertilizer and dP is the (incremental) change in pesticides. Since fertilizer increase is 5 times that of pesticide increase, we will express dF in terms of dP. We then have

$$dC = \frac{4}{1 + F} 5 \; dP + \frac{2}{1 + 2P} dP$$

$$= \left[\frac{20}{1 + F} + \frac{2}{1 + 2P} \right] dP$$

Note: We could have alternatively expressed dP in terms of dF to obtain an equally valid answer to the above question.

3. A firm's profit depends on the reliability of its product, R, and the number of units it sells, Q. The profit function is given by

$$\Pi = 10 \; Q - \frac{Q^2}{R}$$

What is the effect of increasing reliability on profits? What is the effect of increasing reliability on the marginal effect of output increases to the firm's profits? [Hint: the second question involves a cross-partial derivative.]

The effect of increasing R on profits is given by the partial derivative of profits with respect to R. This is given by

$$\frac{\partial \Pi}{\partial R} = \frac{Q^2}{R^2}$$

[Note: To see how this result was obtained, just recall that $1/R = R^{-1}$.]

The second question asks for the second partial derivative

$$\frac{\partial \frac{\partial \Pi}{\partial Q}}{\partial R} = \frac{\partial^2 \Pi}{\partial Q \partial R}$$

We know from the lectures that

$$\frac{\partial^2 \Pi}{\partial Q \partial R} = \frac{\partial^2 \Pi}{\partial R \partial Q}$$

Therefore, the effect of increasing reliability on the marginal effect of output increases on profits is given by the derivative of $\partial \Pi / \partial R$ with respect to output, that is

$$\frac{\partial^2 \Pi}{\partial R \partial Q} = \frac{\partial \left(\frac{Q^2}{R^2} \right)}{\partial Q}$$

$$= 2 \frac{Q}{R^2}$$

Solutions to Problems

1. A firm produces two products, 1 and 2. Suppose its profit function is given by the equation

$$\Pi(q_1, q_2) = 8 + 4q_1q_2 - 4q_1^2 - q_2^2$$

where q_1 and q_2 are the output levels of products 1 and 2, respectively.

a. How does an increase in q_1 affect the firm's profits?

The partial derivative of the profit function with respect to q_1 is

$$\frac{\partial \Pi(q_1, q_2)}{\partial q_1} = 4q_2 - 8q_1.$$

b. How does an increase in q_2 affect the firm's profits?

The partial derivative of the profit function with respect to q_2 is

$$\frac{\partial \Pi(q_1, q_2)}{\partial q_2} = 4q_1 - 2q_2.$$

c. Suppose the value of q_1 is equal to 4. For what levels of output of product 2 does increasing the output of product 1 increase profits?

If the value of q_1 is equal to 4, then, by substituting 4 for q_1 in the answer to part (a), we can see that the partial derivative with respect to q_1 is equal to $4q_2 - 32$.

This is positive if

$$4q_2 - 32 > 0 \quad \Rightarrow$$

$$4q_2 > 32 \quad \Rightarrow$$

$$4q_2 > 32 \quad \Rightarrow$$

$$q_2 > 8.$$

Therefore, an increase in the output level of product 1 from a value of 4 to a slightly higher value increases profits if the output of product 2 is higher than 8.

2. The amount demand for Lexus cars in the US depends on their price and on per capita income (*PCI*), and is given by

$$Q_L = \alpha + \beta\, e^{PCI} - \gamma \frac{P_L}{PCI}$$

where α, β, and γ are positive constants (parameters).

a. What is the effect of an increase in per capita income on the demand for Lexus cars?

The derivative of the demand for Lexus with respect to *PCI* is

$$\frac{\partial Q_L}{\partial PCI} = \beta\, e^{PCI} + \gamma \frac{P_L}{PCI^2}$$

b. Is the effect of a price increase on the demand stronger or weaker if per capita income increases? Show your answers using calculus.

First we must calculate the expression for effect of the price on the demand. This is given by

$$\frac{\partial Q_L}{\partial P_L} = -\frac{\gamma}{PCI}$$

Now, we can readily observe that the higher the value of *PCI*, the smaller the (negative) effect of a price increase on demand.

3. A firm's production costs are a function of the quantity, *Q*, it produces and the product's reliability, *R*. The cost function is given by

$$C = 10\, R^3\, Q^2$$

a. What is the effect of increasing output on production costs?

This is given by the partial derivative of cost with respect to Q, which is

$$\frac{\partial C}{\partial Q} = 20 \; R^3 \; Q$$

b. What is the effect of increasing reliability on costs? Is increasing reliability costly to the firm?

This effect is given by the partial derivative of cost with respect to R, which is

$$\frac{\partial C}{\partial R} = 30 \; R^2 \; Q^2$$

This derivative is clearly positive, since neither reliability nor output can be negative (or zero). Thus, increasing reliability is costly to the firm.

c. What is the effect of increasing reliability on the marginal cost of output? Is the cost of increasing reliability higher for firms that produce more units?

This can be obtained by taking the partial derivative of the answer in part (a) with respect to R, or in other words, by calculating the cross partial derivative of cost with respect to Q and R. This is given by

$$\frac{\partial \frac{\partial C}{\partial Q}}{\partial R} = \frac{\partial \; (20 \; R^3 \; Q)}{\partial \; R} = 60 \; R^2 \; Q$$

This is positive, as both R and Q are positive. Thus, the cost of increasing reliability is higher for firms that produce more units (recall that $\frac{\partial^2 C}{\partial Q \partial R} = \frac{\partial^2 C}{\partial R \partial Q}$).

4. A firm produces two types of products, A and B. The total production cost of the firm as a function of the quantities of A and B it produces is given by:

$$C(Q_A, Q_B) = 3 \; Q_A^2 + Q_B^2 + Q_A \; Q_B$$

This cost function indicates that it is "more expensive" to produce product A than product B (at least if the firm is producing an equal amount of both) and that both products create cost externalities to the production of the other, i.e., the marginal cost if either is increasing in the output of the other.

Suppose the firm can sell a unit of A for a price $P_A = 4$ and a unit of B for a price $P_B = 2$, how much of each product should the firm produce in order to obtain the highest possible profits ? Recall that the firm's profits are equal to total revenue (from both products) minus total cost (of producing both products).

Note: You need only check the First Order conditions. The Second Order conditions are satisfied.

The profits of the firm are given by the function:

$$\Pi(Q_A, Q_B) = P_A\, Q_A + P_B\, Q_B - 3\, Q_A^2 - Q_B^2 - Q_A\, Q_B$$

The First Order Conditions of profit maximization with respect to the choice of output are given by

$$\frac{\partial\, \Pi(Q_A, Q_B)}{\partial\, Q_A} = 0 \quad \Rightarrow \quad P_A - 6\, Q_A - Q_B = 0$$

$$\frac{\partial\, \Pi(Q_A, Q_B)}{\partial\, Q_B} = 0 \quad \Rightarrow \quad P_B - 2\, Q_B - Q_A = 0$$

Substituting in for the prices of A and B we get:

$$4 - 6\, Q_A - Q_B = 0$$

$$2 - 2\, Q_B - Q_A = 0$$

Solving the first equation for Q_B yields

$$Q_B = 4 - 6\, Q_A \tag{1}$$

Substituting this into the second equation yields

$$2 - 8 + 12\, Q_A - Q_A = 0 \quad \Rightarrow$$

$$Q_A = \frac{6}{11}$$

Finally, by plugging this into equation (1) above we get:

$$Q_B = 4 - 6\,\frac{6}{11}$$

$$= \frac{8}{11}$$

We see that despite the fact that the price of product A is higher than that of product B, the firm is producing more of the latter, given that the cost function is "steeper" with respect to A than with respect to B. [Of course, if P_A was sufficiently high the firm would indeed produce more of product A.]

5. A firm produces two types of products, A and B. The total production cost of the firm as a function of the quantities of A and B it produces is given by:

$$C(Q_A, Q_B) = 3\,Q_A^2 + Q_B^2 + Q_A\,Q_B$$

This cost function indicates that it is "more expensive" to produce product A than product B (at least if the firm is producing an equal amount of both) and that both products create cost externalities to the production of the other, i.e., the marginal cost if either is increasing in the output of the other.

If the price of product B is equal to 1 how high does the price of product A have to be before the firm produces an equal amount of both ? Recall that the firm's profits are equal to total revenue (from both products) minus total cost (of producing both products). [Note: the price of product A must treated as an unknown parameter P_A when computing the profits of the firm.]

Denote the price of product A by P_A and the price of product B by P_B . Then, the profits of the firm are given by the function:

$$\Pi(Q_A, Q_B) = P_A\,Q_A + P_B\,Q_B - 3\,Q_A^2 - Q_B^2 - Q_A\,Q_B$$

The First Order Conditions of profit maximization with respect to the choice of output are given by

$$\frac{\partial \Pi(Q_A, Q_B)}{\partial Q_A} = 0 \quad \Rightarrow \quad P_A - 6\, Q_A - Q_B = 0$$

$$\frac{\partial \Pi(Q_A, Q_B)}{\partial Q_B} = 0 \quad \Rightarrow \quad P_B - 2\, Q_B - Q_A = 0$$

Substituting in for $P_B = 1$ we get:

$$P_A - 6\, Q_A - Q_B = 0$$

$$1 - 2\, Q_B - Q_A = 0$$

Solving the first equation for Q_B yields

$$Q_B = P_A - 6\, Q_A \tag{1}$$

Substituting this into the second equation yields

$$1 - 2\, P_A + 12\, Q_A - Q_A = 0 \quad \Rightarrow$$

$$Q_A = \frac{2\, P_A - 1}{11}$$

Finally, by plugging this into equation (1) above we get:

$$Q_B = P_A - 6\, \frac{2\, P_A - 1}{11}$$

$$= \frac{6}{11} - \frac{1}{11}\, P_A$$

For $Q_A = Q_B$ we need:

$$\frac{2\, P_A - 1}{11} = \frac{6}{11} - \frac{1}{11}\, P_A \quad \Rightarrow$$

$$2\, P_A - 1 = 6 - 1\, P_A \quad \Rightarrow$$

$$3\, P_A = 7 \quad \Rightarrow$$

$$P_A = \frac{7}{3}$$

The price of product A has to be more than twice that of product B in order for the firm to produce the same amount of both.

We could obtain the same outcome by using the following short-cut. Consider again the system of First Order Conditions:

$$P_A - 6\ Q_A - Q_B = 0$$

$$1 - 2\ Q_B - Q_A = 0$$

We are looking for a solution in which the two output levels are equal. By setting $Q_A = Q_B$ we then have the system

$$P_A - 6\ Q - Q = 0$$

$$1 - 2\ Q - Q = 0$$

This yields

$$P_A = 7\ Q$$

$$Q = \frac{1}{3}$$

and finally

$$P_A = \frac{7}{3}$$

Tricks likes these greatly simplify the computations. However, one needs to be sure to apply them correctly.